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LIMIT THEOREMS FOR THE INTEGRAL OF A POPULATION PROCESS WITH IMMIGRATION*

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Abstract. A limit theorem is proven for the integral of a general class of population processes possessing independent immigration components. For the special case of the Bellman–Harris process with immigration, further results are obtained.

population process	immigration
branching process with immigration	integral of a process

1. Introduction

In a recent paper, Heyde and Seneta [2] have proven certain limit theorems relating to the total progeny of a discrete-time subcritical branching process with immigration. Their interest was in finding estimates for certain parameters of the process, and their arguments were of the martingale type. Because of this, it is impossible to use their techniques to prove analogous results for the continuous-time model.

The purpose of this note is to provide an alternative approach for dealing with the above problems. In particular, this approach works equally well for the discrete- and continuous-time model both in 1 and p dimensions. Furthermore, more general reproductive mechanisms could also be allowed.

Before we state our results, we will describe the process of interest, $\{Z(t)\}_{t\geq 0}$. It serves our purpose to define a more general process from which the Bellman–Harris branching model with immigration (B.H.I.)

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can be gotten as a special case. Let $\{Y(t)\}_{t\geq 0}$ be any positive stochastic process. Let

$$\mathcal{A} = \{(Y_{ij}(t))_{t\geq 0}\}_{i,j\geq 1}$$

be a collection of independent identically distributed (i.i.d.) copies of the $\{Y(t)\}_{t\geq 0}$ process, and $\{T_i\}_{i\geq 1}$ and $\{v_i\}_{i\geq 1}$ be two independent sequences of i.i.d. random variables which are independent of $\mathcal{A}$. Assume that T_1 is positive w.p. 1 with distribution function $G_0(t)$, and v_1 is integer-valued with p.g.f.

$$f_0(s) = \sum_{j=0}^{\infty} p_0(j)\, s^j\,, \quad |s| \leq 1\,.$$

Define the renewal function by $n(t) = k$ iff $\tau_k \leq t < \tau_{k+1}$, $k \geq 0$, where

$$\tau_0 \equiv 0\,, \qquad \tau_k = \sum_{i=1}^{k} T_i\,, \quad k \geq 1\,.$$

We then put

$$Z(t) = \sum_{i=1}^{n(t)} W_i(t - \tau_i)\,, \quad t \geq 0\,, \tag{1.1}$$

where

$$W_i(t) = \sum_{j=1}^{v_i} Y_{ij}(t)\,, \quad i \geq 1\,,\ t \geq 0\,.$$

Intuitively speaking, the $\{\tau_i\}_{i\geq 1}$ represent the times at which immigrations occur, and the $\{v_i\}_{i\geq 1}$ the numbers of immigrants which appear. Once a particle enters the population, it "grows" according to the $\{Y(t)\}_{t\geq 0}$ process. Furthermore, all particles in the population evolve independently of all other particles.

The above model encompasses a multitude of special processes. For example, if the $\{Y(t)\}$ process is a Bellman–Harris age-dependent process, then the $\{Z(t)\}$ process is a B.H.I. [3]. If $\{Y(t)\}$ is a discrete-time branching process and G_0 is degenerate with all its mass at 1, then $\{Z(t)\}$ is a discrete-time branching process with immigration. The $\{Y(t)\}$ process could just as easily be a generalized age-dependent branching process [4] or a p-dimensional branching process either in discrete or continuous time.

For the remainder of this paper, we assume that $\mathrm{E}\{T_1\} = \lambda_0 < \infty$, $G_0(0+) = 0$ and $p_0(0) < 1$.

Let $\mathcal{G}$ be the class of all positive increasing functions defined on $[0, \infty)$. Define for each $f \in \mathcal{G}$ the random variable

$$A(f) = \lim_{n \to \infty} \left\{ \int_0^{T_1} f(\Sigma_{i=1}^n W_i(u + \Sigma_{j=2}^i T_j))\, \mathrm{d}u \right\}, \tag{1.2}$$

where

$$\sum_{j=2}^{1} T_j = 0 .$$

The limit in (1.2) exists since f is monotone.

We now state our main result.

Theorem 1. *Let $f \in \mathcal{G}$. Then*

$$\lim_{t \to \infty} \left(t^{-1} \int_0^t f(Z(u))\, \mathrm{d}u \right) = \mathbf{E}\{A(f)\}/\lambda_0 \quad w.p.\ 1. \tag{1.3}$$

The limit in (1.3) *holds even if* $\mathbf{E}\{A(f)\} = \infty$.

The proof of Theorem 1 does not depend upon the structure of the $\{Y(t)\}$ process, only upon the fact that immigrants arrive according to a renewal process, and particles behave independent of each other. The $\{Y(t)\}$ process is only relevant in evaluating $\mathbf{E}\{A(f)\}$.

An immediate corollary to Theorem 1 is the following.

Corollary 1.1. *Let $\{f_i\}_{i=1}^k$ be any finite collection of elements of $\mathcal{G}$ such that* $\mathbf{E}\{A(f_i)\} < \infty$, $1 \leqslant i \leqslant k$. *Define*

$$g = \sum_{i=1}^{k} c_i f_i ,$$

where $\{c_i\}$ are arbitrary finite constants. Then (1.3) *holds for g.*

The proof of Corollary 1.1 is immediate from the linearity of the integral.

We now state the results of Heyde and Seneta [2], which, as we shall see, are direct consequences of Corollary 1.1.

Theorem (Heyde and Seneta [2]). *Let* $\{Z(n)\}_{n\geq 0}$ *be a discrete-time branching process with immigration. Let* $f_1(s)$ *be the p.g.f. of the offspring distribution and assume that*

$$m_1 = f_1'(1-) < 1\,, \qquad f_1''(1-) < \infty\,.$$

Assume also that

$$m_0 = f_0'(1-) < \infty\,, \qquad f_0''(1-) < \infty\,.$$

Then, w.p. 1,

$$\lim_{n\to\infty} \left\{ n^{-1} \sum_{i=1}^{n} Z(i) \right\} = \mu = m_0 (1-m_1)^{-1}\,, \tag{1.4}$$

$$\lim_{n\to\infty} \left\{ n^{-1} \sum_{i=1}^{n} (Z(i)-\mu)^2 \right\} = C^2 (1-m_1^2)^{-1}\,, \tag{1.5}$$

where $C^2 = \sigma_0^2 + \sigma_1^2\,\mu$ *with* σ_i^2 *the variance of the distribution given by* $f_i(s)$, $i = 0, 1$.

To prove the above result, all one has to do is evaluate $\mathrm{E}\{A(f)\}$ for the appropriate polynomial f and apply Corollary 1.1. For (1.4), $f(x) = x$, and for (1.5), $f(x) = (x-\mu)^2$. Since the computation of $\mathrm{E}\{A(f)\}$ is straightforward, the details will be omitted.

It turns out that Theorem 1 has an interesting application to Bellman–Harris processes with immigration. Let L be the extinction time for the W process, i.e.

$$L = \inf\{t\colon W(t) = 0\}\,.$$

It has recently been proven [8] that

$$\lim_{t\to\infty} \mathrm{P}[Z(t) = k] = \pi_k\,, \qquad k \geq 0\,, \tag{1.6}$$

with

$$\mathrm{E}\{L\} < \infty \Rightarrow \sum_{k=0}^{\infty} \pi_k = 1\,, \tag{1.7}$$

$$\mathrm{E}\{L\} = \infty \Rightarrow \pi_k = 0\,, \qquad k \geq 0\,. \tag{1.8}$$

Using Theorem 1, we can obtain another interpretation of the π_k and for the case $\mathrm{E}\{L\} < \infty$ we can get an explicit evaluation.

Theorem 2. *Let $\{Z(t)\}_{t\geq 0}$ be a B.H.I. Then:*

(i) $\pi_k = \mathrm{E}\{A(I_k)\}/\lambda_0$, $k \geq 0$, *where I_k is the indicator function of the set $\{k\}$.*

(ii) *Let $N_k[0,t]$ denote the amount of time spent in state k by the Z process up to time t. Then w.p. 1,*

$$\lim_{t\to\infty} \{t^{-1} N_k[0,t]\} = \pi_k , \quad k \geq 0 .$$

The only properties of the B.H.I. that are needed to prove (1.6) and Theorem 2 are that it is integer valued and 0 is an absorbing state. Hence Theorem 2 holds for any Z process satisfying these two conditions.

Our final result deals explicitly with the Bellman–Harris process with immigration. We assume that the lifetime of a particle has distribution function $G_1(t)$, and upon the death of a particle, offspring are produced according to $f_1(s)$. Let

$$I(t) = \int_0^t Z(u)\, \mathrm{d}u .$$

As pointed out by Pakes [6], if $Z(t)$ represents the number of virulent bacteria in a host, then the immigration represents the number of bacteria collected by the host by virtue of its interaction with the habitat. The integral $I(t)$ can be regarded as a measure of the total amount of toxin produced in $[0, t]$.

Applying Theorem 1, it is not difficult to show that

$$\lim_{t\to\infty} \{t^{-1} I(t)\} = m_0\, \lambda_0^{-1}\, \mathrm{E}\{\textstyle\int_0^\infty Y(u)\, \mathrm{d}u\} \quad \text{w.p. 1.} \tag{1.9}$$

A discrete time version of (1.9) can be found in [5].

If $m_1 = 1$, the right side of (1.9) equals ∞. Hence $1/t$ is not the proper normalization. It turns out, as the next theorem shows, that $1/t^2$ is.

Theorem 3. *Assume $m_1 = 1$, $\lambda_1 = \int_0^\infty t\, \mathrm{d}G_1(t) < \infty$, and $\sigma^2 = \frac{1}{2} f_1''(1-) < \infty$. Then $I(t)/t^2$ converges in distribution $(t \to \infty)$ to a random variable I whose distribution has Laplace transform*

$$\mathrm{E}\{\mathrm{e}^{-sI}\} = [\cosh \sqrt{(\tau s)}]^{-\beta} , \tag{1.10}$$

where $\beta = m_0 \lambda_1/\lambda_0 \sigma^2$ and $\tau = \sigma^2/\lambda_1$.

2. Proofs of Theorems 1 and 2

Let

$$J(t) = \int_0^t f(Z(u))\,\mathrm{d}u\,,$$

where $f \in \mathcal{G}$. Using (1.1), it is not difficult to show that

$$J(t) = V(t) + \int_{\tau_{n(t)}}^{t} f(\Sigma_{j=1}^{n(t)} W_j(u-\tau_j))\,\mathrm{d}u\,, \qquad t \geqslant 0,$$

with

$$V(t) = \sum_{i=1}^{n(t)-1} \int_{\tau_i}^{\tau_{i+1}} f(\Sigma_{j=1}^{i} W_j(u-\tau_j))\,\mathrm{d}u\,.$$

The proof of Theorem 1 follows from the next lemma.

Lemma 2.1. *Let $f \in \mathcal{G}$. Then*

$$\lim_{t\to\infty} \{t^{-1} V(t)\} = \mathrm{E}\{A(f)\}/\lambda_0 \qquad \textit{w.p. } 1\,.$$

Proof. A simple change of variable shows

$$V(t) = \sum_{i=1}^{n(t)-1} \int_0^{T_{i+1}} f(\Sigma_{j=1}^{i} W_j(u + \Sigma_{k=j+1}^{i} T_k))\,\mathrm{d}u\,, \tag{2.1}$$

where $\Sigma_{k=i+1}^{i} T_k = 0$, $i \geqslant 1$. Using the monotonicity of f, we can bound $V(t)$ below by

$$B_N(t) = \sum_{j=N}^{n(t)-1} \int_0^{T_{i+1}} f(\Sigma_{j=i-N+1}^{i} W_j(u + \Sigma_{k=j+1}^{i} T_k))\,\mathrm{d}u\,,$$

where N is an arbitrary integer and t is taken large enough so that $n(t) > N+1$. Similarly we can bound $V(t)$ above by

$$U(t) = \sum_{i=1}^{n(t)} \int_0^{T_{i+1}} f(\Sigma_{j=0}^{\infty} W_{i-j}(u + \Sigma_{k=i-j+1}^{i} T_k))\,\mathrm{d}u\,,$$

where the $\{T_i\}_{-\infty<i<\infty}$ are i.i.d. random variables and similarly $\{(W_i(u))_{u\geqslant 0}\}_{-\infty<i<\infty}$ are i.i.d. random processes. Without loss of generality, we can assume that all the random quantities in question are defined on a common probability space.

From the Ergodic Theorem, we conclude that w.p. 1,

$$\liminf_{n(t)\to\infty} [n(t)^{-1} V(t)] \geqslant \lim_{n(t)\to\infty} [n(t)^{-1} B_N(t)]$$

$$= \mathbf{E}\{\textstyle\int_0^{T_1} f(\Sigma_{i=1}^N W_i(u+\Sigma_{j=2}^i T_j))\, du\},$$

$$\limsup_{n(t)\to\infty} [n(t)^{-1} V(t)] \leqslant \lim_{n(t)\to\infty} [n(t)^{-1} U(t)] = \mathbf{E}\{A(f)\}.$$

Since N is arbitrary and $t/n(t) \to \lambda_0$ w.p. 1, the lemma follows. □

We now finish the proof of Theorem 1. Since $J(t) \geqslant V(t)$, we can assume without loss of generality that $\mathbf{E}\{A(f)\} < \infty$. Let $\epsilon, \delta > 0$. By Egorov's Theorem there exists a $T_0 > 0$ such that the set

$$B = \{n(t(1+\epsilon)) > n(t):\ t > T_0\}$$

has probability greater than $1 - \frac{1}{3}\delta$. Furthermore, by Lemma 2.1 and Egorov's Theorem, T_0 can be chosen so large that the set

$$B(T_0) = \{|V(t)/t - \mathbf{E}\{A(f)\}/\lambda_0| < \epsilon:\ t > T_0\}$$

has probability greater than $1 - \frac{1}{3}\delta$. Let $t > T_0(1+\epsilon)$ and consider the set

$$C = B \cap B(T_0) \cap B(T_0(1+\epsilon)).$$

On C,

$$V(t) < J(t) < V(t(1+\epsilon)), \tag{2.2}$$

$$V(t)/t > \mathbf{E}\{A(f)\}/\lambda_0 - \epsilon, \tag{2.3}$$

$$V(t(1+\epsilon))/t < (1+\epsilon)(\mathbf{E}\{A(f)\}/\lambda_0 + \epsilon). \tag{2.4}$$

Theorem 1 follows from (2.2)–(2.4) together with the fact that $\mathbf{P}[C] \geqslant 1 - \delta$. □

We now turn to the proof of Theorem 2. Let I_k^+ be the indicator function of the set $[k,\infty)$, $k \geqslant 0$. Clearly $I_k^+ \in \mathcal{G}$, and so, by Theorem 1,

$$\lim_{t\to\infty} t^{-1} \int_0^t I_k^+(Z(u))\, du = \mathbf{E}\{A(I_k^+)\}/\lambda_0 \quad \text{w.p. } 1. \tag{2.5}$$

However, $I_k = I_k^+ - I_{k+1}^+$. Hence, using the linearity of the integral and (2.5), we obtain

$$\lim_{t\to\infty} \left\{ t^{-1} \int_0^t I_k(Z(u))\,\mathrm{d}u \right\} = \mathbf{E}\{A(I_k)\}/\lambda_0 \qquad \text{w.p. } 1\,. \tag{2.6}$$

Since $t^{-1} \int_0^t I_k(Z(u))\,\mathrm{d}u$ is bounded by 1, we can strengthen the convergence in (2.6) to L^1. Hence

$$\lim_{t\to\infty} \left\{ t^{-1} \int_0^t \mathbf{E}\{I_k(Z(U))\}\,\mathrm{d}u \right\} = \lim_{t\to\infty} \left\{ t^{-1} \int_0^t \mathbf{P}[Z(u)=k]\,\mathrm{d}u \right\}$$

$$= \mathbf{E}\{A(I_k)\}/\lambda_0\,. \tag{2.7}$$

It follows now from (1.6) and (2.7) that $\pi_k = \mathbf{E}\{A(I_k)\}/\lambda_0$. This proves (i) of Theorem 2.

To prove (ii), all we need to note is that

$$\int_0^t I_k(Z(u))\,\mathrm{d}u = N_k[0,t]\,,$$

and substitute in (2.6). This completes the proof of Theorem 2. □

3. Proof of Theorem 3.

Observe that $I(t)$ has the following representation:

$$I(t) = \int_0^t Z(u)\,\mathrm{d}u = \sum_{i=1}^{n(t)} \int_{\tau_i}^t W_i(u-\tau_i)\,\mathrm{d}u = \sum_{i=1}^{n(t)} \int_0^{t-\tau_i} W_i(u)\,\mathrm{d}u\,, \qquad t \geq 0\,. \tag{3.1}$$

Define

$$\psi(s,t) = \mathbf{E}\{\exp[-s \textstyle\int_0^t Y(u)\,\mathrm{d}u] \mid Y(0)=1\}\,,$$

$$H(s,t) = \mathbf{E}\{\exp[-s \textstyle\int_0^t Z(u)\,\mathrm{d}u]\}\,.$$

It follows from (3.1) that

$$H(s,t) = \mathbf{E}\left\{ \prod_{i=1}^{n(t)} f_0(\psi(s,t-\tau_i)) \right\}.$$

The function $\psi(s,t)$ is decreasing in t for s fixed. Furthermore, $\tau_{n(t)} \leqslant t < \tau_{n(t)+1}$ w.p. 1.Hence

$$\mathrm{E}\left\{\prod_{i=1}^{n(t)} f_0(\psi(s,\tau_{n(t)+1}-\tau_i))\right\} \leqslant H(s,t) \leqslant \mathrm{E}\left\{\prod_{i=1}^{n(t)} f_0(\psi(s,\tau_{n(t)}-\tau_i))\right\}. \quad (3.2)$$

Since the $\{T_i\}$ are i.i.d., the collection of random variables $\{\tau_{n(t)}-\tau_i\}_{i=1,\dots,n(t)}$ has the same joint distribution as the collection of random variables $\{\tau_i\}_{i=n(t)-1,\dots,1,0}$. Similarly the collection $\{\tau_{n(t)+1}-\tau_i\}_{i=1,\dots,n(t)}$ has the same joint distribution as $\{\tau_i+T_{n(t)+1}\}_{i=n(t)-1,\dots,1,0}$. (3.2) now becomes

$$\mathrm{E}\left\{\prod_{i=0}^{n(t)-1} f_0(\psi(s,\tau_i+T_{n(t)+1}))\right\} \leqslant H(s,t) \leqslant \mathrm{E}\left\{\prod_{i=0}^{n(t)-1} f_0(\psi(s,\tau_i))\right\}. \quad (3.3)$$

It is well known [1] that for any $L > 0$,

$$\lim_{t\to\infty} \mathrm{P}[T_{n(t)+1} > L] = \lambda_0^{-1}\int_L^\infty u\,\mathrm{d}G_0(u).$$

It follows that for a given $\epsilon > 0$, we can always choose $L > 0$ such that

$$\mathrm{E}\left\{\prod_{i=0}^{n(t)-1} f_0(\psi(s,\tau_i+L))\right\} - \epsilon \leqslant H(s,t) \quad (3.4)$$

for all t sufficiently large. In view of (3.3) and (3.4), it is enough to show that for every $L \geqslant 0$,

$$\lim_{t\to\infty} \mathrm{E}\left\{\prod_{i=1}^{n(t)-1} f_0(\psi(s/t^2,\tau_i+L))\right\} = [\cosh\surd(\tau s)]^{-\beta}. \quad (3.5)$$

To prove (3.5) we will show that

$$\lim_{t\to\infty} \left\{\prod_{i=1}^{n(t)-1} f_0(\psi(s/t^2,\tau_i+L))\right\} = [\cosh\surd(\tau s)]^{-\beta} \text{ w.p. } 1, \quad (3.6)$$

or, equivalently,

$$\lim_{t\to\infty} \left\{-\sum_{i=0}^{n(t)-1} \log f_0(\psi(s/t^2,\tau_i+L))\right\} = \quad (3.7)$$

$$= \beta\surd(s\tau) + \beta\log(\tfrac{1}{2}(1+\exp[-2\surd(s\tau)])).$$

Observe that

$$-\sum_{i=0}^{n(t)-1} \log f_0(\psi(s/t^2, \tau_i + L)) =$$

$$= -\sum_{i=0}^{N} \log f_0(\psi(s/t^2, \tau_i + L)) + m_0 \sum_{i=N+1}^{n(t)-1} \{1 - \psi(s/t^2, \tau_i + L)\}$$

$$+ \sum_{i=N+1}^{n(t)-1} [-\log f_0(\psi(s/t^2, \tau_i + L)) - m_0\{1 - \psi(s/t^2, \tau_i + L)\}]$$

$$= \alpha_1(N, t) + \alpha_2(N, t) + \alpha_3(N, t) .$$

The choice of N will be indicated later.

Let $\epsilon > 0$. For any fixed N,

$$\limsup_{t \to \infty} |\alpha_1(N, t)| = 0 .$$

Consider now $\alpha_2(N, t)$. Since $m_1 = 1$, $\lim_{t \to \infty} \psi(s, t) = \psi(s)$ exists and $\psi(s)$ is the Laplace transform of a legitimate distribution. Put

$$\alpha_2(N, t) = m_0(n(t) - N - 2)\{1 - \psi(s/t^2)\}$$

$$- m_0 \sum_{i=N+1}^{n(t)-1} \{\psi(s/t^2, \tau_i + L) - \psi(s/t^2)\} .$$

Pakes [6] has shown that

$$1 - \psi(s/t^2) \sim t^{-1}\surd(s/\tau) .$$

Hence

$$\lim_{t \to \infty} \{m_0(n(t) - N - 2)(1 - \psi(s/t^2))\} = \beta \surd(s\tau) .$$

We now examine the remaining sum

$$A_N(t) = m_0 \sum_{i=N+1}^{n(t)-1} \{\psi(s/t^2, \tau_i + L) - \psi(s/t^2)\} .$$

Put

$$G_{1\,0}(t) = \begin{cases} 1\,, & t \geqslant 0\,, \\ 0\,, & t < 0\,, \end{cases}$$

$$G_{1\,n}(t) = \int_0^t G_{1\,n-1}(t-y)\,\mathrm{d}G_1(y)\,, \qquad n \geqslant 1\,.$$

Pakes [6] has proven the following proposition.

Proposition 3.1. *For* $0 \leqslant s < 1$, *let*

$$\psi_0(s) \equiv 1\,, \qquad \psi_{n+1}(s) = f_1(\psi_n(s))\,v(s)\,, \qquad n \geqslant 1\,,$$

where

$$v(s) = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}G_1(t)\,.$$

Then for $t \geqslant 0$, $n \geqslant 1$,

$$\begin{aligned} \psi_n(s) - \psi(s) - (1-\psi(s))\,G_{1\,n}(t) &\leqslant \psi(s,t) - \psi(s) \\ &\leqslant \psi_n(s) - \psi(s) + (1-\mathrm{e}^{-st}\,\psi(s))\,(1-G_{1\,n}(t))\,. \end{aligned} \tag{3.9}$$

Set

$$m(i) = [(1-\epsilon)\,\lambda_0\,i/\lambda_1]\,,$$

where $[x]$ is the greatest integer less than x. Using the right-side inequality in (3.9), we obtain

$$\begin{aligned} A_N(t) \leqslant m_0 &\sum_{i=N+1}^{n(t)-1} \{\psi_{m(i)}(s/t^2) - \psi(s/t^2)\} \\ &+ m_0 \sum_{i=N+1}^{n(t)-1} \{(1-\exp[-s(\tau_i+L)/t^2]\,\psi(s/t^2)) \\ &\qquad \times (1-G_{1\,m(i)}(\tau_i+L))\} \\ = S_1(t) &+ S_2(t)\,. \end{aligned}$$

The Weak Law of Large Numbers implies that for any $\epsilon > 0$,

$$\limsup_{t\to\infty} |S_2(t)| < \epsilon\,, \tag{3.10}$$

provided N is sufficiently large. We now examine $S_1(t)$. Pakes [6] has shown that $\psi_n(s) - \psi(s)$ has upper and lower bounds of the form

$$D(s)(1-U(s))\,\frac{\varphi(s)\,U^n(s)}{1+\varphi(s)\,U^n(s)}\,,$$

where

$$D(s) \to 1/\sigma^2\,, \qquad \varphi(s) \to 1\,, \qquad 1-U(s) \sim 2\sqrt{(\sigma^2\lambda_1 s)} \qquad \text{as } s \to 0\,.$$

Using the above facts it is not difficult to show that

$$\limsup_{t\to\infty} |S_1(t) + \beta \log(\tfrac{1}{2}(1+\exp[-2\sqrt{(s\tau)}])| < \epsilon\,, \tag{3.11}$$

again provided N is sufficiently large. (3.10) and (3.11) imply that

$$\limsup_{t\to\infty} A_N(t) \leqslant -\beta\log(\tfrac{1}{2}(1+\exp[-2\sqrt{(s\tau)}]) + 2\epsilon\,, \tag{3.12}$$

provided N is sufficiently large.

In a similar way, if we use the left-side inequality of the proposition, we can show that

$$\liminf_{t\to\infty} A_N(t) \geqslant -\beta\log(\tfrac{1}{2}(1+\exp[-2\sqrt{(s\tau)}]) - 2\epsilon\,, \tag{3.13}$$

for N sufficiently large.

From (3.12) and (3.13) we conclude that

$$\limsup_{t\to\infty} |\alpha_2(N,t) - \{\beta\sqrt{(s\tau)} + \beta\log(\tfrac{1}{2}(1+\exp[-2\sqrt{(s\tau)}])\}| < \epsilon\,, \tag{3.14}$$

provided N is sufficiently large.

It remains to deal with $\alpha_3(N,t)$. From the Mean-Value Theorem, we have for any $\delta > 0$,

$$|1 - \log f_0(s) - m_0(1-s)| \leqslant \delta(1-s)$$

provided s is close enough to 1. As already noted, $\lim_{t\to\infty}\psi(s,t) \downarrow \psi(s)$ with $\psi(s)$ the Laplace Transform of a legitimate randon variable. Hence $\psi(s/t^2, \tau_i + L) \geqslant \psi(s/t^2)$ for all i, and $\psi(s/t^2) \to 1$ as $t \to \infty$.

Therefore, for N fixed,

$$\limsup_{t\to\infty} |\alpha_3(N,t)| \leqslant \delta \limsup_{t\to\infty} |\alpha_2(N,t)| .$$

Since we have already shown that $\limsup_{t\to\infty} |\alpha_2(N,t)|$ is bounded and δ is arbitrary, we conclude that for any fixed N,

$$\limsup_{t\to\infty} |\alpha_3(N,t)| = 0 .$$

Thus (3.7) follows from (3.8), (3.14) and (3.15). This completes the proof of Theorem 3. □

Remark. The same kind of argument can be used to find the limit distribution for the total number of particles born in $[0,t]$, or the total number of deaths in $[0,t]$.

Note. A generalization of Theorem 3 has recently been proven by Pakes [7]. He assumes that the immigration times (τ_i) satisfy the conditions

(a) the (τ_i) are increasing with w.p. 1;
(b) $\lim_{i\to\infty}\{i^{-1}\tau_i\}$ exists and is positive w.p. 1.

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